

Can the Brans-Dicke Gravity with Λ Possibly be a Theory of Dark Matter?

Hongsu Kim*

International Center for Astrophysics,

Korea Astronomy and Space Science Institute, Daejeon 305-348, Korea

Abstract

The pure Brans-Dicke (BD) gravity with or without the cosmological constant Λ has been taken as a model theory for the dark matter. Indeed, there has been a consensus that unless one modifies either the standard theory of gravity, namely, general relativity, or the standard model for particle physics, or both, one can never achieve a satisfying understanding of the phenomena associated with dark matter and dark energy. Along this line, our dark matter model in this work can be thought of as an attempt to modify the gravity side alone in the simplest fashion to achieve the goal. Among others, it is demonstrated that our model theory can successfully predict the emergence of dark matter halo-like configuration in terms of a self-gravitating spacetime solution to the BD field equations and reproduce the flattened rotation curve in this dark halo-like object in terms of the non-trivial energy density of the BD scalar field, which was absent in the context of general relativity where Newton's constant is strictly a "constant" having no dynamics. Our model theory, however, is not entirely without flaw, such as the prediction of relativistic jets in all types of galaxies which actually is not the case.

* e-mail: hongsu@kasi.re.kr

I. INTRODUCTION

There has been a consensus among researchers that unless one modifies either the standard theory of gravity, namely, general relativity, or standard particle physics theory (say, the Weinberg-Glashow-Salam's standard model), or both, one can never achieve a satisfying understanding of the phenomena associated with dark matter and dark energy. Along this line, in the present work, we would like to propose an attempt to modify the gravity side alone in the simplest fashion to achieve the goal. To be more specific, the pure Brans-Dicke (BD) gravity [1] with or without the cosmological constant Λ shall be taken as a model theory for the dark matter. Indeed, the BD theory is the most studied, and hence the best-known, of all the alternative theories of classical gravity to Einstein's general relativity [2]. This theory can be thought of as a minimal extension of general relativity designed to properly accomodate both Mach's principle [2, 3] and Dirac's large number hypothesis [2, 3]. Namely, the theory employs the viewpoint in which Newton's constant G is allowed to vary with space and time and can be written in terms of a scalar ("BD scalar") field as $G = 1/\Phi$. Besides, the BD scalar field (and the BD theory itself) is not of quantum origin. Rather, it is classical in nature and hence can be expected to serve as a very relevant candidate to play some role in the late-time evolution of the universe, such as the present epoch.

As a scalar-tensor theory of gravity, the BD theory involves an adjustable, but undetermined, "BD-parameter" ω , and as is well-known, the larger the value of ω , the more dominant the tensor (curvature) degree and the smaller the value of ω , the larger the effect of the BD scalar. Also as long as we select a sufficiently large value of ω , the predictions of the theory agree perfectly with all the observations/experiments to date [2]. For this reason, the BD theory has remained a viable theory of classical gravity. However, no particularly overriding reason has ever emerged to take it seriously over general relativity. As shall be presented shortly in this work, here we emphasize that it is the expected existence of dark matter (and dark energy as well, see Ref.4) that may put the BD theory over general relativity as a more relevant theory of classical gravity consistent with observations that have so far been unexplained within the context of general relativity.

As mentioned above, in our model theory for dark matter, the effect of the cosmological constant Λ shall be generally considered. Here, Λ is essentially supposed to play the role of dark energy in the context of the BD theory as has been studied in detail in Ref.4. As

is well-known, the mysterious *flattened* rotation curves observed for so long in the outer regions of galactic halos have been the primary cause that called for the existence of dark matter. Among others, therefore, we shall demonstrate in this work that our model theory can successfully predict the emergence of a dark matter halo-like configuration in terms of a self-gravitating static and nearly spherically-symmetric spacetime solution to the BD field equations and reproduce the flattened rotation curve in the outer region of this dark halo-like object in terms of the non-trivial energy density of the BD scalar field, which is absent in the context of general relativity where Newton’s constant is strictly a “constant” having no dynamics.

II. SCHWARZSCHILD-DE SITTER-TYPE SOLUTION IN THE BD THEORY OF GRAVITY

As stated earlier in the introduction, we would like to demonstrate in the present work that the BD gravity with or without the cosmological constant can reproduce some representative features of dark matter, such as the formation of a dark matter halo inside of which the flattened rotation curves are observed. Since the galactic dark matter halos are roughly static and spherically-symmetric, we should, among others, look for such dark matter halo-like solution to the BD field equations. Thus, in this section, we shall construct the Schwarzschild-de Sitter-type solution in the BD theory and claim later on that it can represent the dark matter halo in the context of our model of dark matter.

As is well-known, even in Einstein’s general relativity, Finding exact solutions to the highly non-linear Einstein field equations is a formidable task. For this reason, algorithms generating exact, new solutions from the known solutions of simpler situations have been actively looked for, and actually quite a few have been found. In the BD theory of gravity, the field equations are even more complex; thus, it is natural to seek similar algorithms generating exact solutions from the already known simpler solutions either of the BD theory or of the conventional Einstein gravity. In particular, Tiwari and Nayak [5] proposed an algorithm that allows stationary axisymmetric solutions in the vacuum BD theory to be generated from the known Kerr solution in the vacuum Einstein theory. Thus, in the

present work, we shall take the algorithm suggested by Tiwari and Nayak [5] or by Singh and Rai [6] to construct the Schwarzschild-de Sitter-type solution in the BD theory in the presence of the cosmological constant from the well-known Schwarzschild-de Sitter solution in Einstein gravity with the cosmological constant.

Consider the BD theory in the presence of the (positive) cosmological constant Λ described by the action

$$S = \int d^4x \sqrt{g} \left[\frac{1}{16\pi} \left(\Phi R - \omega \frac{\nabla_\alpha \Phi \nabla^\alpha \Phi}{\Phi} \right) - \Lambda \right], \quad (1)$$

where Φ is the BD scalar field representing roughly the inverse of Newton's constant and ω is a generic parameter of the theory. Extremizing this action with respect to the metric $g_{\mu\nu}$ and the BD scalar field Φ yields the classical field equations given, respectively, by

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{8\pi}{\Phi}\Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}^{BD}, \quad \nabla_\alpha \nabla^\alpha \Phi = -\frac{32\pi}{(2\omega + 3)}\Lambda \quad \text{where} \\ T_{\mu\nu}^{BD} &= \frac{1}{8\pi} \left[\frac{\omega}{\Phi^2} (\nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2}g_{\mu\nu} \nabla_\alpha \Phi \nabla^\alpha \Phi) + \frac{1}{\Phi} (\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \nabla_\alpha \nabla^\alpha \Phi) \right]. \end{aligned} \quad (2)$$

The Einstein gravity is the $\omega \rightarrow \infty$ limit of this BD theory. Now the algorithm of Tiwari and Nayak [5] or Singh and Rai [6] goes as follows; In general, let the metric for a stationary axisymmetric solution to the Einstein field equations take the form

$$ds^2 = -e^{2U_E} (dt + W_E d\phi)^2 + e^{2(k_E - U_E)} [(dx^1)^2 + (dx^2)^2] + h_E^2 e^{-2U_E} d\phi^2 \quad (3)$$

while letting the metric for a stationary axisymmetric solution to the BD field equations be

$$ds^2 = -e^{2U_{BD}} (dt + W_{BD} d\phi)^2 + e^{2(k_{BD} - U_{BD})} [(dx^1)^2 + (dx^2)^2] + h_{BD}^2 e^{-2U_{BD}} d\phi^2 \quad (4)$$

where U , W , k , and h are functions of x^1 and x^2 only. The significance of the choice of the metric in this form has been thoroughly discussed by Matzner and Misner [7] and Misra and Pandey [8]. Tiwari and Nayak or Singh and Rai first wrote down the Einstein and the BD field equations for the choice of metrics in Eqs. (3) and (4), respectively. Comparing the two sets of field equations closely, then, they realized that stationary axisymmetric solutions of the BD field equations are obtainable from those of the Einstein field equations provided certain relations between metric functions hold. That is, if (W_E, k_E, U_E, h_E) form a stationary axisymmetric solution to the Einstein field equations for the metric in Eq. (3),

then a corresponding stationary axisymmetric solution to the BD field equations for the metric in Eq. (4) is given by $(W_{BD}, k_{BD}, U_{BD}, h_{BD}, \Phi)$, where

$$\begin{aligned} W_{BD} &= W_E, \quad k_{BD} = k_E, \quad U_{BD} = U_E - \frac{1}{2} \log \Phi, \\ h_{BD} &= [h_E]^{(2\omega-1)/(2\omega+3)}, \quad \Phi = [h_E]^{4/(2\omega+3)}. \end{aligned} \quad (5)$$

Now what remains is to apply this method to obtain the Schwarzschild-de Sitter-type solution in the BD theory from the known Schwarzschild-de Sitter solution in Einstein gravity. To do so, one needs some preparation, which involves casting the Schwarzschild-de Sitter solution given in the usual Schwarzschild coordinates (t, r, θ, ϕ) in the metric form given in Eq. (3) by performing the coordinate transformation (of r alone) suggested by Misra and Pandey [8]. Namely, we start with

$$\begin{aligned} ds^2 &= - \left(1 - \frac{2M}{r} - \frac{8\pi\Lambda}{3} r^2 \right) dt^2 + \left(1 - \frac{2M}{r} - \frac{8\pi\Lambda}{3} r^2 \right)^{-1} dr^2 + r^2 [d\theta^2 + \sin^2 \theta d\phi^2] \\ &= - \frac{\Delta}{r^2} dt^2 + r^2 \left[\frac{dr^2}{\Delta} + d\theta^2 + \sin^2 \theta d\phi^2 \right], \end{aligned} \quad (6)$$

where $\Delta = r^2(1 - 8\pi\Lambda r^2/3) - 2Mr$ with M and Λ denoting the ADM mass and the (positive) cosmological constant, respectively. Consider now such a transformation of the radial coordinate as the one introduced by Misra and Pandey [8];

$$r = r(R) \quad \text{such that} \quad \frac{dr^2}{\Delta} = dR^2. \quad (7)$$

Then, the Schwarzschild-de Sitter solution can now be cast in the form in Eq. (3), i.e.,

$$ds^2 = - \frac{\Delta}{L^2} dt^2 + L^2 [dR^2 + d\theta^2] + L^2 \sin^2 \theta d\phi^2 \quad (8)$$

with $L \equiv r(R)$ and, hence, $\Delta = L^2(1 - 8\pi\Lambda L^2/3) - 2ML$. Now, we can read off the metric components as

$$e^{2U_E} = \frac{\Delta}{L^2}, \quad W_E = 0, \quad e^{2k_E} = \Delta, \quad h_E^2 = \Delta \sin^2 \theta. \quad (9)$$

Then, using the rule in Eq. (5) in the algorithm by Tiwari and Nayak or by Singh and Rai, one finds that the metric components of the Schwarzschild-de Sitter-type solution in the BD theory come out to be

$$\begin{aligned} e^{2U_{BD}} &= \frac{\Delta}{L^2} (\Delta \sin^2 \theta)^{-2/(2\omega+3)}, \quad W_{BD} = 0, \quad e^{2k_{BD}} = \Delta, \\ h_{BD}^2 &= (\Delta \sin^2 \theta)^{(2\omega-1)/(2\omega+3)}, \quad \Phi(R, \theta) = (\Delta \sin^2 \theta)^{2/(2\omega+3)}. \end{aligned} \quad (10)$$

Next, looking at the forms of the metric for a stationary axisymmetric solution to the Einstein or the BD field equations given in eqs. (3) and (4), one can realize that the factors in the metric functions,

$$e^{2(k-U)}, \quad W^2 e^{2U}, \quad h^2 e^{-2U} \quad (11)$$

should have length dimension 2. Firstly, for the solution to the Einstein field equations,

$$e^{2(k_E-U_E)} = L^2, \quad W_E^2 e^{2U_E} = 0, \quad h_E^2 e^{-2U_E} = L^2 \sin^2 \theta \quad (12)$$

which all take the right length dimension, 2, as expected. Secondly, for the solution to the BD field equations, however, we have

$$\begin{aligned} e^{2(k_{BD}-U_{BD})} &= L^2 \left(\Delta \sin^2 \theta \right)^{2/(2\omega+3)}, \quad W_{BD}^2 e^{2U_{BD}} = 0, \\ h_{BD}^2 e^{-2U_{BD}} &= L^2 \sin^2 \theta \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)}. \end{aligned} \quad (13)$$

Thus, in order for this metric solution to the BD field equations to have the right length dimension, it turns out that the factor $(\Delta \sin^2 \theta)^{\pm 2/(2\omega+3)}$ should be dimensionless. Therefore, in order to guarantee this, we introduce a normalization factor r_0 to render the factor Δ (particularly appearing in $(\Delta \sin^2 \theta)^{\pm 2/(2\omega+3)}$) dimensionless, i.e., $\Delta = [L^2(1 - 8\pi\Lambda L^2/3) - 2ML]/r_0^2$. The proper value of this normalization factor r_0 shall be characterized later on when we do the numerics to quantify the rotation velocity in our model.

Then, by transforming back to the standard Schwarzschild coordinates using Eq. (7), we finally arrive at the Schwarzschild-de Sitter-type solution in the Brans-Dicke theory given by

$$\begin{aligned} ds^2 &= \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \left[- \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3} r^2 \right) c^2 dt^2 + r^2 \sin^2 \theta d\phi^2 \right] \\ &\quad + \left(\Delta \sin^2 \theta \right)^{2/(2\omega+3)} \left[\left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3} r^2 \right)^{-1} dr^2 + r^2 d\theta^2 \right], \\ \Phi(r, \theta) &= \frac{1}{G_0} \left(\Delta \sin^2 \theta \right)^{2/(2\omega+3)}, \end{aligned} \quad (14)$$

where we have restored both the present value of Newton's constant G_0 and the speed of light c in order to come from the geometrized unit ($G_0 = c = 1$) back to CGS

units. That is, we have replaced M with $G_0 M/c^2 \equiv \tilde{M}$ along with $\Lambda G_0 \Lambda/c^4 \equiv \tilde{\Lambda}$; hence, $\Delta = [r^2(1 - 8\pi\tilde{\Lambda}r^2/3) - 2\tilde{M}r]/r_0^2$. A remarkable feature of this Schwarzschild-de Sitter-type solution is the fact that, unlike the Schwarzschild-de Sitter solution in general relativity, the spacetime it describes is static (i.e., non-rotating), but *not* spherically-symmetric. Note also that as $\omega \rightarrow \infty$, this Schwarzschild-de Sitter-type solution goes over to the standard Schwarzschild-de Sitter solution in Einstein gravity as it should since the $\omega \rightarrow \infty$ limit of the BD theory is the Einstein gravity. Note also that the Brans-Dicke-Schwarzschild (BDS) spacetime solution to these vacuum BD field equations has been found some time ago [9] in rather a theoretical attempt to construct non-trivial black hole spacetime solutions in the BD theory. There the BDS spacetime solution has been obtained by setting $a = e = 0$ in the Brans-Dicke-Kerr-Newman (BDKN) solution in Eq. (11) of Ref.9, and it coincides with the Schwarzschild-type solution that results when we set $\tilde{\Lambda} = 0$ in the solution constructed above, as it should.

Now coming back to our main objective, we are interested in the role played by the BD scalar field as dark matter, particularly in forming galactic dark matter halos inside of which the well-known rotation curves have been observed. We suggest (and later on demonstrate in terms of the reproduction of a flat rotation curve) that, indeed, the BD scalar field can successfully play the role of dark matter. In order for this to happen, we first claim that the BD scalar can cluster into a halo-like configuration as it can be represented by this Schwarzschild-de Sitter-type solution constructed above in Eq. (14). At this point, therefore, it seems relevant to address the nature of potential singularities of this Schwarzschild-de Sitter-type metric solution. Just as the Schwarzschild-de Sitter solution in general relativity, it appears to possess two coordinate singularities which would arise at $\Delta = [r^2(1 - 8\pi\tilde{\Lambda}r^2/3) - 2\tilde{M}r]/r_0^2 = 0$; (i) the inner Schwarzschild gravitational radius, and (ii) the outer de Sitter radius. Generally, it is well-known that the Schwarzschild metric solution in static coordinates possesses an event horizon, that the solution essentially describes the region *outside* this event horizon, and that its inside is not everywhere well-defined. Meanwhile, the de Sitter metric solution in static coordinates has a cosmological horizon, the solution mainly describes the region *inside* this cosmological horizon, and its outside is not everywhere well-defined. As such, the Schwarzschild-de Sitter metric solution in static coordinates is able to represent the region between the inner event horizon and the

outer cosmological horizon, but *not* elsewhere. In the same spirit, our Schwarzschild-de Sitter-type metric solution in the BD theory given in static coordinates has been adopted here *only* to represent the galactic dark halo region, but not below (i.e., the interior of a given galaxy) nor beyond (i.e., the scale of galaxy clusters or even of entire universe). Therefore, all we have to do is to demonstrate that if we quantify the locations of these two singularities by putting real numbers, they are well below and beyond the typical domain of galactic halos.

First, we start with (i) the inner Schwarzschild gravitational radius. For small- r , $\Delta = 0$ is approximated by $r - 2\tilde{M} \simeq 0$ whose solution is $r \simeq 2G_0M/c^2 \simeq 0.01 \text{ pc}$. Here we used, for the typical (total) mass of a galaxy, $M \sim 10^{11}M_\odot$. Obviously, this occurs well inside a galaxy as the typical size of a galaxy ranges from few kpc (for dwarf galaxies) to few hundred kpc (for ordinary galaxies). Indeed, the physical meaning of this singularity is that if the entire galaxy is squeezed into this gravitational radius, it becomes a black hole with its event horizon placed at 0.01 pc . Namely, for ordinary galaxies, this inner Schwarzschild gravitational radius is a *failed* Schwarzschild event horizon. Next, we turn to (ii) the outer de Sitter radius. For large- r , $\Delta = 0$ is approximated by $r - 8\pi\tilde{\Lambda}r^3/3 \simeq 0$, whose solution is $r \simeq (8\pi G_0\Lambda/3c^4)^{-1/2} \simeq 4 \text{ Gpc}$. Here we used for the cosmological constant the observed value, $\Lambda \simeq 10^{-8} \text{ erg/cm}^3$. Apparently, this occurs well beyond a single galaxy; Indeed, this is the scale of entire universe (given the age of the universe, that is $\tau \sim 13.7 \text{ Gyr}$, its rough size would be $c\tau \sim 4 \text{ Gpc}$). In other words, this outer de Sitter radius is totally irrelevant.

To summarize, the potential singularities of the Schwarzschild-de Sitter-type solution are irrelevant to keeping us from employing the solution to represent the galactic dark halo region and, hence, are harmless. Next, the seeming angular singularity at $\theta = 0, \pi$ are irrelevant as well because the domain of principal interest is the neighborhood of galactic equatorial plane, $\theta = \pi/2$, inside the halo where most gases and stars orbit around with flat rotation curves. Normally, the symmetry axis is the last thing to be expected to be singular. Indeed, close inspection reveals that the metric function factor $(\Delta \sin^2 \theta)^{\pm 2/(2\omega+3)}$ is generated by the solution-generating algorithm of Tiwari and Nayak [5] or of Singh and Rai [6] which takes the (singularity-free) solutions of the Einstein field equations to those of the BD field equations. That is, since the appearance of the factor $(\sin^2 \theta)^{\pm 2/(2\omega+3)}$, which is responsible for the singular nature of the symmetry axis, results *simply* from the solution-generating algorithm, one naturally might expect that the symmetry axis $\theta = 0, \pi$ cannot possibly be a

genuine curvature singularity as it would not really represent, say, an infinite concentration of matter along there. However, the symmetry axis $\theta = 0, \pi$, indeed, appears to be singular as the invariant curvature polynomial, such as the curvature scalar R , which can be readily calculated by contracting the metric field equation and using the field equation for the BD scalar field in Eq. (2), is given by

$$R = \frac{4\omega}{(2\omega + 3)^2} \frac{1}{r^2 \Delta} (\Delta \sin^2 \theta)^{-2/(2\omega+3)} \left[r_0^2 \Delta'^2 + 4 \left\{ (2\omega + 3) 4\pi \Lambda G_0 r^2 + \cot^2 \theta \right\} \Delta \right],$$

where $\Delta' = 2r \left[1 - (\tilde{M}/r) - (16\pi \tilde{\Lambda} r^2/3) \right] / r_0^2$, blows up there for generic ω values. Namely, the symmetry axis appears to be a real singularity rather than a coordinate singularity (that can be removed by a change of coordinates). This certainly is a very unexpected and hence puzzling feature of the Schwarzschild-de Sitter-type solution in BD gravity theory. Therefore, one might wonder what would happen to matter habiting near the galactic poles when it comes close to the pole. We now hope to clarify the true nature of this peculiar singularity along the symmetry axis in some detail.

Among others, the first thing that comes to our mind regarding the nature of this singularity along the symmetry axis is the fact that, unlike the familiar curvature singularity at the center of a black hole, which is point-like, this singularity is an infinitely-extended line singularity. This means that perhaps the fate of matter coming close to this singularity would be somewhat different from what one would normally expect for matter approaching a point-like black hole singularity. To get right to the point, it turns out that in the *immediate vicinity* of the symmetry axis, the specific energy (i.e., the energy per unit mass) of a test particle becomes extremely high. As a result, the particle moves along the symmetry axis at nearly the ultrarelativistic speed (i.e., at nearly the speed of light). Of course, this is an unusual feature, which has no analogue in the Einstein gravity context, and a close inspection reveals that it can eventually be attributed to the metric function factor $(\Delta \sin^2 \theta)^{\pm 2/(2\omega+3)}$, which is responsible for the singular nature of the symmetry axis. The explicit demonstration of this peculiar behavior of test particles near the symmetry axis shall be presented later on in section V in terms of a rigorous analysis of the geodesic motion there.

In conclusion, this study of the true nature of the singularity along the symmetry axis leads us to suspect that perhaps the bizzare singularity at $\theta = 0, \pi$ of the Schwarzschild-de Sitter-type solution in BD gravity theory can account for the relativistic *bipolar outflows* (*twin jets*) extending from the central region of “active galactic nuclei (AGNs)” Namely,

the curious singularity along the symmetry axis seems harmless, after all. Instead, it turns out to be a pleasant surprise as it can explain a long-known puzzle in observed features of galaxies. A cautious comment might be needed here, though. That is, the relativistic bipolar outflows have been observed only for some particular types of galaxies, such as AGNs and micro-quasars. (Radio galaxies, Seyfert galaxies and quasars fall into the AGN category.) Namely, the jets do not seem to be a general feature of all types of galaxies.

Lastly, the failure of asymptotic flatness of this Schwarzschild-de Sitter-type solution is not of serious concern here as we essentially aim at finding a spacetime solution that can represent the dark halo-like configuration, which is known to cluster only on a sub- Mpc scale, i.e., a local scale, and eventually matches the cosmological geometry, such as the Friedmann-Robertson-Walker metric for homogeneous and isotropic expanding universe, at a larger cosmological scale in order for the BD theory in the presence of the cosmological constant to provide a successful model for dark matter and dark energy, as suggested, for example, in Ref.4.

Our natural next mission is then to ask whether these configurations really can reproduce the properties of dark matter halos, namely, if our BD scalar model for dark matter can reproduce the flattening of the rotation velocity curves inside these halo configurations consistent with the observations. Therefore, in the following, we shall address this issue; first, for the case when only the dark matter content represented by the BD scalar is present, i.e., when the dark matter halo is represented by the Schwarzschild-type solution (without the cosmological constant (Λ) term), and then for the other case when both the dark matter content and the dark energy content, represented by the cosmological constant, are present, i.e., when the dark halo is represented by the Schwarzschild-de Sitter-type solution (with the Λ term).

III. BD DARK MATTER HALOS NEGLECTING THE DARK ENERGY (Λ) CONTRIBUTION: THE CASE OF THE JORDAN FRAME

As claimed earlier, it appears that the BD scalar can indeed cluster into a halo-like configuration as it can be represented by the Schwarzschild-type solution in the BD theory. Thus, we now attempt to obtain the rotation curves in our BD scalar halo and eventually to

demonstrate that they are actually flattened far out to the distant region of halo. First of all, since we need concrete “numbers” we now restore both Newton’s constant G_0 and the speed of light c in order to come from the geometrized unit ($G_0 = c = 1$) back to the CGS unit. Then, the energy-momentum tensor of the BD scalar field given earlier in Eq. (2) should now be multiplied by the factor (c^4/G_0) , and in the Schwarzschild-type solution in eEq. (14) above, we should replace M by $G_0 M/c^2 \equiv \tilde{M}$, as we mentioned earlier. Apparently then, the energy-momentum tensor of the BD scalar field with restored G_0 and c has the dimension of the energy-momentum density in CGS units (erg/cm^3).

We now turn to the computation of energy density profile and (anisotropic) pressure components of the BD scalar field playing the role of the dark matter by treating the BD scalar field as a (dark matter) *fluid*. The BD scalar field fluid, however, would fail to be a “perfect” fluid as can readily be envisaged from the fact that the associated BDS solution configuration is not spherically-symmetric. Namely, its pressure cannot be “isotropic”, i.e., $P_r \neq P_\theta \neq P_\phi$. Such a fluid may be called an *imperfect* fluid due to the *anisotropic* pressure components, and as such, its stress tensor can be written as

$$T_\nu^{BD \mu} = \begin{pmatrix} -c^2\rho & 0 & 0 & 0 \\ 0 & P_r & T_\theta^r & 0 \\ 0 & T_r^\theta & P_\theta & 0 \\ 0 & 0 & 0 & P_\phi \end{pmatrix}, \quad (15)$$

which is to be contrasted to its counterpart for the usual perfect fluid with isotropic pressure given by the well-known form $T_\nu^\mu = P\delta_\nu^\mu + (c^2\rho + P)U^\mu U_\nu = diag(-c^2\rho, P, P, P)$, where $U^\alpha = dX^\alpha/d\tau$ (with τ being the proper time) denotes the 4-velocity of the fluid element normalized such that $U^\alpha U_\alpha = -c^2$. Note that in addition to the diagonal entries representing the (anisotropic) pressure components $T_i^i = P_i$. (with no sum over i), there are off-diagonal entries T_θ^r, T_r^θ representing a *shear stress* that also results from the failure of spherical symmetry. It is also interesting to note that a stress tensor of this sort (given in eEq. (15)) arises in the case of rotating boson star made up of a complex scalar field [11]. Thus by substituting the BDS solution given in Eq. (14) into the BD energy-momentum tensor in eq.(2) and then setting it equal to Eq.(15), we can eventually read off the energy density and the pressure components of the BD scalar field imperfect fluid;

$$\rho = \frac{c^2}{8\pi G_0} \frac{4}{(2\omega + 3)^2} \frac{1}{r^2 \tilde{\Delta}} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)}$$

$$\begin{aligned}
& \times \left[2(\omega + 1) \left\{ (r - \tilde{M})^2 + \tilde{\Delta} \cot^2 \theta \right\} - (2\omega + 3) \tilde{M}(r - \tilde{M}) \right], \\
P_r &= -\frac{c^4}{8\pi G_0} \frac{4}{(2\omega + 3)^2} \frac{1}{r^2 \tilde{\Delta}} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \\
& \times \left[(r - \tilde{M})^2 + (2\omega + 3) \tilde{M}(2\tilde{M} - r) + 2(\omega - 1) \tilde{\Delta} \cot^2 \theta \right], \\
P_\theta &= \frac{c^4}{8\pi G_0} \frac{4}{(2\omega + 3)^2} \frac{1}{r^2 \tilde{\Delta}} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \left[(2\omega + 3)(r - \tilde{M})(r - 2\tilde{M}) \right. \\
& \left. - 2(\omega - 1)(r - \tilde{M})^2 + \left\{ 2(\omega + 1) \cos^2 \theta - (2\omega + 3) \right\} \frac{\tilde{\Delta}}{\sin^2 \theta} \right], \\
P_\phi &= -\frac{c^4}{8\pi G_0} \frac{4}{(2\omega + 3)^2} \frac{1}{r^2 \tilde{\Delta}} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \\
& \times \left[2(\omega + 1)(r - \tilde{M})^2 - \tilde{\Delta} \cot^2 \theta - (2\omega + 3)(r - \tilde{M})(r - 2\tilde{M}) \right], \\
T_\theta^r &= \Delta T_r^\theta = \frac{c^4}{8\pi G_0} \frac{4}{(2\omega + 3)^2} \frac{1}{r^2} \cot \theta \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \\
& \times \left[4\omega(r - \tilde{M}) - (2\omega + 3)(r - 2\tilde{M}) \right],
\end{aligned} \tag{16}$$

where $\Delta = [r(r - 2\tilde{M})]/r_0^2$ and $\tilde{\Delta} = r_0^2 \Delta$. Note that the off-diagonal components T_θ^r and T_r^θ are *odd* functions of θ while the diagonal components $(\rho, P_r, P_\theta, P_\phi)$ are *even* functions of the polar angle under $\theta \rightarrow (\pi - \theta)$. As a result, the off-diagonal components vanish (i.e., no shear stress survives) if we average over this polar angle to get a net stress. Thus, the equation of state of this BD scalar k-essence fluid forming a galactic halo is given by

$$w = \frac{P}{c^2 \rho} = -\frac{\left[(r - \tilde{M})^2 + (2\omega + 3) \tilde{M}(2\tilde{M} - r) + 2(\omega - 1) \tilde{\Delta} \cot^2 \theta \right]}{2(\omega + 1) \left\{ (r - \tilde{M})^2 + \tilde{\Delta} \cot^2 \theta \right\} - (2\omega + 3) \tilde{M}(r - \tilde{M})}, \tag{17}$$

where $P = P_r$. Namely, $P = w(r, \theta) c^2 \rho$ with $w(r, \theta) \sim O(1)$, meaning that this BD scalar fluid is essentially a *barotropic* fluid but with a “position-dependent” coefficient $w(r, \theta)$. Note that although the BD scalar field is a candidate for dark matter, it is not quite a dust. In principle, the speed of sound in this BD scalar field fluid can also be evaluated via $c_s^2 = dP/d\rho$, but we shall not discuss that in any more detail in this work.

We are now ready to compute the behavior of the rotation curves in the outer region (i.e., at large, but finiter, say, $r \gg G_0 M/c^2$) of our BD scalar field halo. To be more precise, for a galaxy of typical (total) mass $M \sim 10^{11} M_\odot$, the outer region of its dark matter halo, say, $r \sim 10$ (kpc) $\simeq 10^{23}$ (cm) is much greater than $G_0 M/c^2 \simeq 10^{16}$ (cm) by a factor of 10^7 or so. Thus, to this end, we first approximate the expressions for its energy density and the (radial) pressure given in Eq. (16) for large- r ;

$$\rho \simeq \frac{c^2}{8\pi G_0} \frac{8(\omega + 1)}{(2\omega + 3)^2} \frac{1}{r^2 \sin^2 \theta} \Delta^{-2/(2\omega+3)} \sin^{-4/(2\omega+3)} \theta, \tag{18}$$

$$P \simeq -\frac{c^4}{2\pi G_0} \frac{1}{(2\omega + 3)^2} \frac{1}{r^2} \left[2(\omega - 1) \cot^2 \theta + 1 \right] \Delta^{-2/(2\omega+3)} \sin^{-4/(2\omega+3)} \theta.$$

Note that in the above approximations and in the discussions below, it was and it shall be assumed that the metric function $\Delta = [r(r - 2\tilde{M})]/r_0^2 \simeq (r/r_0)^2$ for larger. It is interesting to note that as a candidate for dark matter, the energy density ρ of the BD scalar field is almost certainly *positive everywhere* (i.e., for both small and larger). In the meantime, its (radial) pressure P particularly on a larger scale (i.e., for larger) turns out to be *negative* although its sign appears unclear on a small scale (i.e., for small r).

Finally, we are ready to determine the rotation curve inside our BD scalar field halo. First, we start by recalling the origin of the long-standing puzzle associated with the galaxy rotation curves. In the most naive sense, the apparent rotation velocity of an object at a radius r from the galactic center would be given in the Newtonian limit by Kepler's third law, $v^2 = G_0 M(r)/r$. Indeed, inside a given galaxy where the luminous mass is nearly uniformly distributed, the rotation velocity has been observed to grow roughly linearly with the distance r , consistent with this Kepler's law. In the outer region of a galactic halo, however, the rotation velocity is expected to behave like $v \sim 1/\sqrt{r}$ as the luminous mass is confined within the extent of the given galaxy. The observations, however, exhibit *flattened* rotation curves, and this has been the age-old dilemma that called for the existence of dark matter in the outer regions of galactic halos.

In the present work, therefore, we shall employ the expression for the rotation velocity still given by Kepler's law, but instead attempt to explain the flattened rotation curve in terms of the non-trivial energy density of the BD scalar field, which appears to play the role of dark matter. Namely, the non-trivial mass density of the BD scalar field turns out to contribute to the mass function $M(r)$ in such a way as to render the rotation curve flat, as we shall see in a moment.

At this point, regarding the construction of the rotation velocity, we have some comments. For the present case, it may seem that one cannot just employ Kepler's law as the stress tensor of the BD scalar field (playing the role of dark matter) and that the background Schwarzschild-de Sitter-type spacetime in the BD theory is not spherically

symmetric. Nevertheless, gases or stars orbiting around the host galaxy center usually lie on the galactic equatorial plane. This implies that the orbital motions on the equatorial plane $\theta = \pi/2$ are of particular interest; hence, there one may still wish to employ Kepler's law. Along this line, therefore, one might wish to employ the relativistic counterpart to this rotation velocity equation, instead. Later on in the appendix, the construction of the rotation velocity shall be promoted to a fully relativistic version in terms of the rigorous derivation of timelike geodesics in the Schwarzschild-de Sitter-type spacetime. The quantitative results that we are about to present below, however, essentially remain unchanged even for the fully relativistic treatment.

Now for our case, using the BD scalar energy density profile given earlier, we have $M(r) = \int_0^{2\pi} d\phi \int_\epsilon^{\pi-\epsilon} d\theta \int_0^r dr \sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}}\rho(r, \theta) = (2c^2/G_0) [(\omega + 1)/(2\omega + 1)(2\omega + 3)] f(\omega)r(r/r_0)^{-2/(2\omega+3)}$ and hence,

$$v^2(r) = \frac{G_0 M(r)}{r} = c^2 \frac{2(\omega + 1)}{(2\omega + 1)(2\omega + 3)} f(\omega) \left(\frac{r}{r_0} \right)^{-2/(2\omega+3)}, \quad (19)$$

where $f(\omega) \equiv \int_\epsilon^{\pi-\epsilon} d\theta \sin^{-[1+2/(2\omega+3)]} \theta = 2 \int_0^{1-\delta} dx [1-x^2]^{-(2\omega+4)/(2\omega+3)}$ with $\epsilon, \delta \ll 1$. (Note here that the integration over the polar angle θ starts not from 0 but from $\epsilon \ll 1$ as the symmetry axis $\theta = 0$ of the BDS solution in Eq. (14) possesses the danger of an internal infinity nature, namely, the symmetry axis is an infinite proper distance away, as discussed carefully in Ref.10.) It has been known for some time that in order for the BD theory to remain a viable theory of classical gravity passing all the observational/experimental tests to date, the BD ω -parameter has to have a large value, say, $|\omega| \geq 500$ [2]. In our previous study [9], however, we realized that the static solution to the vacuum BD field equations given in Eq. (14) above, but without Λ , can turn into a black hole spacetime for $-5/2 \leq \omega < -3/2$. Thus, now for $|\omega| \geq 500$, the same static solution Eq. (14) we are considering represents just a halo-like configuration with a regular geometry everywhere (i.e., having no horizon), which is static but not exactly spherically symmetric (note that the galactic halos are also believed to be nearly spherically symmetric, but not exactly). Thus, if we substitute a large- ω value, say, $\omega \sim 10^6$ into Eq. (19) above, evidently $M(r) \sim r$; hence, we get

$$v(r) \simeq 100(km/s) \times \left(\frac{r}{r_0} \right)^{-(1/10^6)} \quad (20)$$

because for $\omega \sim 10^6$, $f(\omega) \simeq O(1)$. Namely, for this large- ω value, the rotation curve gets flattened out as $r^{-(10^{-6})} \sim \text{constant}$, and its magnitude becomes several hundred km/s .

Here, it is particularly remarkable that the observationally and experimentally allowed large value of $\omega \sim 10^6$ renders the rotation curve flat and fixes its magnitude to several hundred km/s at the *same* time! This is, indeed, an attractive feature of the BD theory that distinguishes it from other theories of dark matter proposed thus far. Of course, this is in impressive agreement with the data for rotation curves observed in spiral/elliptic galaxies with $M/L \simeq (10 - 20)M_\odot/L_\odot$ and in low-surface-brightness (LSB)/dwarf galaxies with $M/L \simeq (200 - 600)M_\odot/L_\odot$ (where M/L denotes the so-called “mass-to-light” ratio given in the units of the solar mass-to-luminosity ratio and exhibiting a large excess of dark matter over the luminous matter) [12]. For instance, the rotation curve of the dwarf spiral galaxy M33 is shown in Fig.1. Rotation curves are observed usually via measurements of the

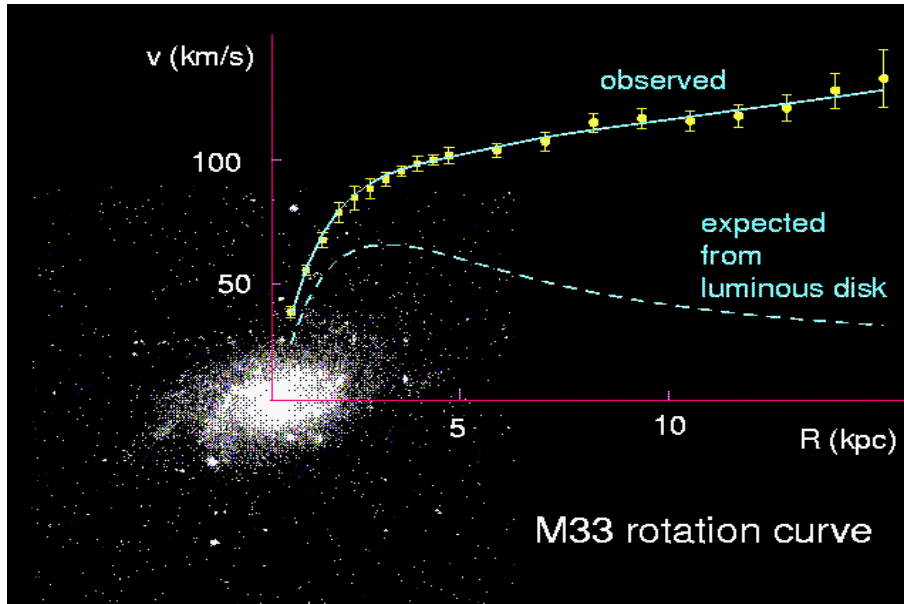


FIG. 1: Flattened rotation curve in the dark halo of dwarf spiral galaxy M33.

Doppler shift of the $21 - cm$ emission line from neutral hydrogen (HI) for distant galaxies and of the light emitted by stars of nearby galaxies [13, 14]. It is also interesting to note that this behavior of the rotation curve in our BD theory dark matter halo model is *independent* of the mass of the host galaxy. Namely, this behavior of the rotation curve comes exclusively from the nature of the dark matter, i.e., the BD scalar field. Indeed, we have restricted our interest in the present work only to the contribution coming from the dark matter component, i.e., the BD scalar field and *not* to the combined contributions from both the luminous and the dark matter. Therefore, the resulting galactic rotation curve (having a

contribution coming *only* from the BD scalar, which is the dark matter candidate) comes out as being independent of the luminous mass in each galaxy. The well-known Tully-Fisher relation [15] between the *total* luminosity of a given galaxy and the outermost rotation velocity can be accounted for if both the luminous and the dark matter contributions are considered. This issue has not been addressed in the present work, but hopefully it should be answered by any dark matter model in order for the proposed model to be truly successful.

We also point out that even if we employ more exact expression for the rotation velocity curve involving the Doppler shift of light emitted by the orbiting objects (assuming that the k-essence halo is almost sphericallysymmetric), namely $v^2(r) = G_0 M(r)/r + 4\pi r^2 G_0 P/c^2$ [16] (with P being the radial pressure given in Eq. (18) above), the conclusions above remain the same. Thus, in what follows, we shall demonstrate this in some detail. Using Eqs. (18) and (19), we have

$$\begin{aligned} v^2(r) &= \frac{G_0 M(r)}{r} + \frac{4\pi G_0}{c^2} r^2 P \\ &= c^2 \frac{2(\omega+1)}{(2\omega+1)(2\omega+3)} f(\omega) \left(\frac{r}{r_0}\right)^{-2/(2\omega+3)} - c^2 \frac{2[2(\omega-1)\cot^2\theta+1]}{(2\omega+3)^2} \sin^{-4/(2\omega+3)}\theta \left(\frac{r}{r_0}\right)^{-4/(2\omega+3)}. \end{aligned} \quad (21)$$

Again, due to the failure of spherical symmetry in the BDS solution, Eq.(14), which is identified with a dark matter halo configuration, this rotation velocity comes out to be polar angle (θ) dependent. However since the gases or the stars orbiting around the galaxy center usually lie on the galactic (equatorial) plane, the relevant situation to consider is the case $\theta = \pi/2$ in which

$$\begin{aligned} v(r) &= c \left[\frac{2(\omega+1)}{(2\omega+1)(2\omega+3)} f(\omega) \left(\frac{r}{r_0}\right)^{-2/(2\omega+3)} - \frac{2}{(2\omega+3)^2} \left(\frac{r}{r_0}\right)^{-4/(2\omega+3)} \right]^{1/2} \\ &\simeq 100(km/s) \times \left(\frac{r}{r_0}\right)^{-(1/10^6)}, \end{aligned} \quad (22)$$

where again we have chosen the value $\omega \sim 10^6$ in the last line.

Next, the equation of state in eEq. (17) of this BD scalar field fluid becomes, in the outer region of the galactic dark matter halo (i.e., at larger),

$$w \simeq - \frac{[2(\omega-1)\cos^2\theta + \sin^2\theta]}{2(\omega+1)} \quad (23)$$

which is obviously negative due to the *negative* pressure (and still *positive* energy density) in this outer region. Moreover, for a large- ω value, i.e., $\omega \sim 10^6$, for which the rotation

curve gets flattened out, that we just have realized, this equation of state at larger further approaches $w \simeq -\cos^2 \theta \simeq -O(1)$. (Incidentally, it is interesting to note that in the vicinity of the equatorial plane $\theta = \pi/2$, $w = 0$; namely, the BD scalar behaves nearly as dust.) This observation is particularly interesting as it appears to indicate that the BD scalar field we are considering possesses *dark -energy-like* negative pressure on larger scales. This observation is indeed consistent with our previous study [4], which showed that on a cosmological scale, the BD scalar field, which has been identified with a “k-essence” there, did exhibit the nature of dark energy possessing a negative pressure.

IV. BD DARK MATTER HALOS INCLUDING THE DARK ENERGY (Λ) CONTRIBUTION

By substituting the Schwarzschild-de Sitter-type solution given in Eq. (14),, but this time into the *total* energy-momentum tensor including the contribution coming from the cosmological constant term

$$\begin{aligned} T_{\mu\nu} &= T_{\mu\nu}^{BD} - \frac{1}{G_0\Phi}\Lambda g_{\mu\nu} \\ &= \frac{c^4}{8\pi G_0} \left[\frac{\omega}{\Phi^2} (\nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \Phi \nabla^\alpha \Phi) + \frac{1}{\Phi} (\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \nabla_\alpha \nabla^\alpha \Phi) \right] - \frac{1}{G_0\Phi} \Lambda g_{\mu\nu}, \end{aligned} \quad (24)$$

and then again setting it equal to Eq.(15), we can eventually read off the energy density and the pressure components of the BD scalar field imperfect fluid as

$$\begin{aligned} \rho &= \frac{c^2}{8\pi G_0} \frac{4}{(2\omega+3)^2} \frac{1}{r^2 \tilde{\Delta}} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \left[\left\{ (r - \tilde{M}) - \frac{16\pi}{3} \tilde{\Lambda} r^3 \right\} \right. \\ &\quad \times \left[2\omega \left\{ (r - \tilde{M}) - \frac{16\pi}{3} \tilde{\Lambda} r^3 \right\} + \left\{ 2(r - \tilde{M}) - (2\omega+3)\tilde{M} + (2\omega-1)\frac{8\pi}{3} \tilde{\Lambda} r^3 \right\} \right] \\ &\quad \left. + 2(\omega+1)\tilde{\Delta} \cot^2 \theta \right] + \frac{(2\omega-1)}{(2\omega+3)} \frac{\Lambda}{c^2} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)}, \\ P_r &= -\frac{c^4}{8\pi G_0} \frac{4}{(2\omega+3)^2} \frac{1}{r^2 \tilde{\Delta}} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \left[\left\{ (r - \tilde{M}) - \frac{16\pi}{3} \tilde{\Lambda} r^3 \right\} \right. \\ &\quad \times \left[2(\omega+1) \left\{ (r - \tilde{M}) - \frac{16\pi}{3} \tilde{\Lambda} r^3 \right\} + \left\{ 2(r - \tilde{M}) - (2\omega+3)\tilde{M} + (2\omega-1)\frac{8\pi}{3} \tilde{\Lambda} r^3 \right\} \right] \\ &\quad \left. - (2\omega+3)\tilde{\Delta}(1 - 16\pi \tilde{\Lambda} r^2) + 2(\omega-1)\tilde{\Delta} \cot^2 \theta \right] - \frac{(2\omega-1)}{(2\omega+3)} \Lambda \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)}, \\ P_\theta &= \frac{c^4}{8\pi G_0} \frac{4}{(2\omega+3)^2} \frac{1}{r^2 \tilde{\Delta}} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \\ &\quad \times \left[\left\{ (r - \tilde{M}) - \frac{16\pi}{3} \tilde{\Lambda} r^3 \right\} \left[(2\omega+3) \frac{\tilde{\Delta}}{r} - 2(\omega-1) \left\{ (r - \tilde{M}) - \frac{16\pi}{3} \tilde{\Lambda} r^3 \right\} \right] \right] \end{aligned}$$

$$\begin{aligned}
& + \left\{ 2(\omega + 1) \cos^2 \theta - (2\omega + 3) \right\} \frac{\tilde{\Delta}}{\sin^2 \theta} \Big] - \frac{(2\omega - 1)}{(2\omega + 3)} \Lambda \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)}, \\
P_\phi &= -\frac{c^4}{8\pi G_0} \frac{4}{(2\omega + 3)^2} \frac{1}{r^2 \tilde{\Delta}} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \\
& \times \left[\left\{ (r - \tilde{M}) - \frac{16\pi}{3} \tilde{\Lambda} r^3 \right\} \left[2(\omega + 1) \left\{ (r - \tilde{M}) - \frac{16\pi}{3} \tilde{\Lambda} r^3 \right\} - (2\omega + 3) \frac{\tilde{\Delta}}{r} \right] \right. \\
& \left. - \tilde{\Delta} \cot^2 \theta \right] - \frac{(2\omega - 1)}{(2\omega + 3)} \Lambda \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)}, \\
T_\theta^r &= \Delta T_r^\theta = \frac{c^4}{8\pi G_0} \frac{4}{(2\omega + 3)^2} \frac{1}{r^2} \cot \theta \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \\
& \times \left[4\omega \left\{ (r - \tilde{M}) - \frac{16\pi}{3} \tilde{\Lambda} r^3 \right\} - (2\omega + 3) \frac{\tilde{\Delta}}{r} \right]
\end{aligned} \tag{25}$$

where now $\Delta = [r^2(1 - 8\pi\tilde{\Lambda}r^2/3) - 2\tilde{M}r]/r_0^2$ and $\tilde{\Delta} = r_0^2\Delta$. As in the previous case where the contribution from the dark energy component, namely, Λ has been neglected, the off-diagonal components T_θ^r , T_r^θ are *odd* functions of θ whereas the diagonal components $(\rho, P_r, P_\theta, P_\phi)$ are *even* functions of the polar angle under $\theta \rightarrow (\pi - \theta)$. Consequently, the off-diagonal components vanish (i.e., no shear stress survives) if we average over this polar angle to get a net stress. Again, first the equation of state of this BD scalar k-essence fluid forming a galactic halo is given by

$$w = \frac{P}{c^2 \rho}. \tag{26}$$

Equations (25) and (26) indicate that $P = w(r, \theta)c^2\rho$ with $w(r, \theta) \sim O(1)$ meaning that this BD scalar fluid is still a *barotropic* fluid but with “position-dependent” coefficient $w(r, \theta)$.

As before, we now turn to the exploration of the behavior of rotation curves in the outer region (i.e., at large but finite- r , say, $r \sim 10$ (kpc) $\simeq 10^{23}$ (cm) $\gg G_0 M/c^2 \simeq 10^{16}$ (cm)) of our BD scalar field halo. And to this end, we first approximate the expressions for the energy density and the (radial) pressure of the BD scalar fluid given in eq.(25) using essentially $\Delta = [r^2(1 - 8\pi\tilde{\Lambda}r^2/3) - 2\tilde{M}r]/r_0^2 \simeq [r^2(1 - 8\pi\tilde{\Lambda}r^2/3)]/r_0^2$ for large- r . They are

$$\begin{aligned}
\rho &\simeq \frac{c^2}{8\pi G_0} \frac{8(\omega + 1)}{(2\omega + 3)^2} \frac{1}{r^2 \sin^2 \theta} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \\
& + \left[\frac{(2\omega - 1)}{(2\omega + 3)} - \frac{4}{3} \frac{(2\omega + 1)}{(2\omega + 3)^2} \right] \frac{\Lambda}{c^2} \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)}, \\
P &\simeq -\frac{c^4}{2\pi G_0} \frac{1}{(2\omega + 3)^2} \frac{1}{r^2} \left[2(\omega - 1) \cot^2 \theta + 1 \right] \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \\
& - \left[\frac{(2\omega - 1)}{(2\omega + 3)} + \frac{4}{3} \frac{1}{(2\omega + 3)^2} \right] \Lambda \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)}.
\end{aligned} \tag{27}$$

It is interesting to note that the energy density ρ of the BD scalar field even in the presence of $\tilde{\Lambda}$, the dark energy component, is almost certainly *positive everywhere* (i.e., for both small and large- r). In the mean time, its (radial) pressure P , in the presence of $\tilde{\Lambda}$, particularly at larger scale (i.e., for large- r) turns out to be *negative*.

We are now ready to determine the rotation curve inside our BD scalar field halo. First, using the BD scalar field energy density profile given earlier, we are supposed to compute the *mass function* $M(r) = \int_0^{2\pi} d\phi \int_{\epsilon}^{\pi-\epsilon} d\theta \int_0^r dr \sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}} \rho(r, \theta)$ in which we have $\sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}} = (\Delta \sin^2 \theta)^{1/(2\omega+3)} \Delta^{-1/2} (r^3/r_0) \sin \theta$. Although we shall consistently employ the approximation $\Delta \simeq [r^2(1 - 8\pi\tilde{\Lambda}r^2/3)]/r_0^2$ as we are interested in the behavior of rotation curves in the outer region (i.e., at large- r) of the BD scalar halo, the actual computation of the mass function $M(r)$ is unfortunately not available since the analytic integration over r cannot be done in a closed form. Indeed one of our main objectives in this section, where the contribution from the dark energy component (i.e., Λ) has been included, involves the theoretical derivation of, say, the upper bound on the value of the cosmological constant Λ that would result in, despite its presence, the nearly flattened rotation curves as we observe them. Obviously, the (positive) cosmological constant has positive contribution to the mass-energy density ρ and as a result, in the presence of Λ , the mass function $M(r)$ grows *non-linearly* with the distance (i.e., the radius of the dark halo) ruining the asymptotic flatness of the rotation curves. As such, in order for our model, i.e., the BD theory with (positive) cosmological constant to successfully reproduce the flattened rotation curves, it is evident that the absolute magnitude of the cosmological constant should be small enough. For this reason, we shall, for the sake of explicit, analytic computation of the mass function $M(r)$, assume that the dimensionless quantity, $\tilde{\Lambda}r^2$ be very small, i.e., $(G_0\Lambda/c^4)r^2 \ll 1$ and see what we would end up with. Then we have

$$\begin{aligned}
M(r) &\simeq \int_0^{2\pi} d\phi \int_{\epsilon}^{\pi-\epsilon} d\theta \int_0^r dr \left\{ \frac{c^2}{8\pi G_0} \frac{8(\omega+1)}{(2\omega+3)^2} \frac{r}{\sin \theta} + \left[\frac{(2\omega-1)}{(2\omega+3)} - \frac{4}{3} \frac{(2\omega+1)}{(2\omega+3)^2} \right] \frac{\Lambda}{c^2} r^3 \sin \theta \right\} \\
&\times \Delta^{-1/2} (\Delta \sin^2 \theta)^{-1/(2\omega+3)} \\
&\simeq \frac{c^2}{G_0} \frac{2(\omega+1)}{(2\omega+3)^2} f(\omega) \\
&\times \left[\frac{(2\omega+3)}{(2\omega+1)} r + \frac{(2\omega+5)}{(6\omega+7)} \frac{4\pi G_0}{3c^4} \Lambda r^3 + \frac{2}{(10\omega+13)} \left(\frac{4\pi G_0}{3c^4} \Lambda \right)^2 r^5 \right] \left(\frac{r}{r_0} \right)^{-2/(2\omega+3)} \\
&+ 2\pi \left\{ \frac{(2\omega-1)}{(2\omega+3)} - \frac{4(2\omega+1)}{3(2\omega+3)^2} \right\} \frac{\Lambda}{c^2} g(\omega)
\end{aligned} \tag{28}$$

$$\times \left[\frac{(2\omega+3)}{(6\omega+7)} r^3 + \frac{(2\omega+5)}{(10\omega+13)} \frac{4\pi G_0}{3c^4} \Lambda r^5 + \frac{2}{(14\omega+19)} \left(\frac{4\pi G_0}{3c^4} \Lambda \right)^2 r^7 \right] \left(\frac{r}{r_0} \right)^{-2/(2\omega+3)}$$

where now $f(\omega) \equiv \int_{\epsilon}^{\pi-\epsilon} d\theta \sin^{-[1+2/(2\omega+3)]} \theta = 2 \int_0^{1-\delta} dx [1-x^2]^{-(2\omega+4)/(2\omega+3)} \simeq O(1)$ and $g(\omega) \equiv \int_{\epsilon}^{\pi-\epsilon} d\theta \sin^{[1-2/(2\omega+3)]} \theta = 2 \int_0^{1-\delta} dx [1-x^2]^{-1/(2\omega+3)} \simeq O(1)$ with $\epsilon, \delta \ll 1$. And here we used $\Delta^{-1/2} \Delta^{-1/(2\omega+3)} \simeq (1+4\pi G_0 \Lambda r^2/3c^4) (1+8\pi G_0 \Lambda r^2/3(2\omega+3)c^4) r^{-1} (r/r_0)^{-2/(2\omega+3)}$ since we assumed $(G_0 \Lambda/c^4) r^2 \ll 1$ as explained above. Lastly, the rotation velocity is given by

$$\begin{aligned} v^2(r) &= \frac{G_0 M(r)}{r} = c^2 \frac{2(\omega+1)}{(2\omega+1)(2\omega+3)} f(\omega) \left(\frac{r}{r_0} \right)^{-2/(2\omega+3)} \\ &+ \left[\left\{ \left[\frac{(2\omega-1)}{(6\omega+7)} - \frac{4(2\omega+1)}{3(2\omega+3)(6\omega+7)} \right] \frac{2\pi G_0}{c^2} \Lambda g(\omega) + \frac{2(\omega+1)(2\omega+5)}{(2\omega+3)^2(6\omega+7)} \frac{4\pi G_0}{3c^2} \Lambda f(\omega) \right\} r^2 \right. \\ &+ c^2 \left\{ \frac{4(\omega+1)}{(2\omega+3)^2(10\omega+13)} \left(\frac{4\pi G_0}{3c^4} \Lambda \right)^2 f(\omega) \right. \\ &+ \left. \left[\frac{(2\omega-1)}{(2\omega+3)} - \frac{4(2\omega+1)}{3(2\omega+3)^2} \right] \frac{(2\omega+5)}{(10\omega+13)} \frac{3}{2} \left(\frac{4\pi G_0}{3c^4} \Lambda \right)^2 g(\omega) \right\} r^4 \\ &\left. + c^2 \left\{ \left[\frac{(2\omega-1)}{(2\omega+3)} - \frac{4(2\omega+1)}{3(2\omega+3)^2} \right] \frac{3}{(14\omega+19)} \left(\frac{4\pi G_0}{3c^4} \Lambda \right)^3 g(\omega) \right\} r^6 \right] \left(\frac{r}{r_0} \right)^{-2/(2\omega+3)}. \end{aligned} \quad (29)$$

Thus again if we substitute a large- ω value, say, $\omega \sim 10^6$ into eq.(29) above, we get, as $f(\omega), g(\omega) \sim O(1)$,

$$\begin{aligned} v^2(r) &\simeq \left[100(km/s) \times \left(\frac{r}{r_0} \right)^{-(1/10^6)} \right]^2 \\ &+ \left[\frac{2\pi}{3} \left(\frac{G_0}{c^2} \Lambda r_0^2 \right) \left(\frac{r}{r_0} \right)^2 + c^2 \frac{3}{10} \left(\frac{4\pi G_0}{3c^4} \Lambda r_0^2 \right)^2 \left(\frac{r}{r_0} \right)^4 \right] \left(\frac{r}{r_0} \right)^{-(1/10^6)}. \end{aligned} \quad (30)$$

where we introduced the normalization factor $r_0 \sim 10 \text{ (kpc)} \simeq 10^{23} \text{ (cm)}$ which represent a typical distance to the outer region of the dark halo where rotation curves begin to get flattened. We are now in a position to *theoretically* determine the upper bound on the value of the (positive) cosmological constant in our dark matter model, namely the BD theory in the presence of the cosmological constant. That is, in order for this result in eq.(30) to successfully describe the flattened rotation curve in the outer region of dark halo, $r \geq r_0$, it should be imposed that

$$\begin{aligned} (A) \quad & \frac{G_0 \Lambda}{c^2} r_0^2 \ll (100 km/s)^2, \\ (B) \quad & \left(\frac{G_0 \Lambda}{c^4} r_0^2 \right)^2 \ll 10^{-6} \end{aligned}$$

as $[G_0\Lambda/c^2] = 1/s^2$ and $[G_0\Lambda/c^4] = 1/cm^2$. Then using $G_0 = 6.67 \times 10^{-8} (cm^3/gs^2)$ and $c = 3 \times 10^{10} (cm/s)$, the conditions (A) and (B) give $\Lambda \ll 1.35 \times 10^{-4} (erg/cm^3)$. It thus is interesting to remark that this *theoretical* upper bound on the value of the cosmological constant is consistent with the observed value $\Lambda_{obs} \simeq 10^{-8} (erg/cm^3)$ (or $\Lambda_{obs}/c^2 \simeq 10^{-29} (g/cm^3)$) [17, 18]. Of course an alternative interpretation is acceptable as well. That is, with the observed small value $\Lambda_{obs} \simeq 10^{-8} (erg/cm^3)$, our model for dark matter turns out to be able to reproduce the flattened rotation curve in a successful manner. Therefore, it appears that our model, based on the Brans-Dicke theory in the presence of the cosmological constant, can successfully reproduce both the dark matter halo configuration and the flattened rotation curves inside of it. This may indicate that our model could be one of the promising candidates for dark matter but other known evidences for dark matter need to be tested as well in this context of BD theory (possibly with cosmological constant) in order for it to be truly successful model of dark matter (and dark energy).

Next, the equation of state in eq.(26) of this BD scalar field becomes, in the outer region of the galactic dark matter halo (i.e., at large- r),

$$\begin{aligned} w &\simeq -\frac{\left[2(\omega-1)\cos^2\theta+\sin^2\theta\right]+[(2\omega-1)(2\omega+3)+4/3](2\pi G_0/c^4)\Lambda r^2\sin^2\theta}{2(\omega+1)+[(2\omega-1)(2\omega+3)-4(2\omega+1)/3](2\pi G_0/c^4)\Lambda r^2\sin^2\theta} \\ &\simeq -\frac{[(2\omega-1)(2\omega+3)+4/3]}{[(2\omega-1)(2\omega+3)-4(2\omega+1)/3]} \end{aligned} \quad (31)$$

which is obviously negative due to the *negative* pressure (and still *positive* energy density) in this outer region. Moreover, for the large- ω value, i.e., $\omega \sim 10^6$ for which the rotation curve gets flattened out, this equation of state at large- r further approaches $w \simeq -1$. Once again, this observation is particularly interesting as it appears to indicate that the BD scalar we are considering possesses *dark energy-like* negative pressure on larger, cosmological scales consistently with our previous study [4] that on the cosmological scale, the BD scalar field with Λ does exhibit the nature of dark energy possessing the negative pressure.

V. RIGOROUS RELATIVISTIC TREATMENT OF THE ROTATION VELOCITY AND THE BIPOLAR JETS

A. The Rotation Velocity - a Fully Relativistic Derivation

In the previous section, the rotation velocity of a test body orbiting in the background of the Schwarzschild-de Sitter-type spacetime in BD theory has been taken as the expression in the Newtonian limit approximation. In the present section, the construction of the rotation velocity shall be promoted to a fully relativistic version in terms of the rigorous derivation of timelike geodesics in the Schwarzschild-de Sitter-type spacetime.

The Schwarzschild-de Sitter-type spacetime in BD theory given in eq.(14) in the text is static and axisymmetric. Thus it possesses time-translational isometry generated by the timelike Killing vector $\xi^\mu = (\partial/\partial t)^\mu = \delta_t^\mu$ and rotational isometry generated by the axial Killing vector $\psi^\mu = (\partial/\partial \phi)^\mu = \delta_\phi^\mu$. And the associated conserved quantities are energy $\tilde{E} = E/m$ and angular momentum $\tilde{L} = L/m$ (per unit rest mass) of a test body along the geodesic

$$-\tilde{E} = u_\mu \xi^\mu = g_{\alpha t} u^\alpha = - \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3} r^2 \right) c^2 \left(\frac{dt}{d\tau} \right), \quad (32)$$

$$\tilde{L} = u_\mu \psi^\mu = g_{\alpha \phi} u^\alpha = \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} r^2 \sin^2 \theta \left(\frac{d\phi}{d\tau} \right) \quad (33)$$

where $u^\mu = (dx^\mu/d\tau)$ denotes the 4-velocity which is tangent to the geodesic. In addition, we have $g_{\mu\nu} u^\mu u^\nu = -\kappa$, namely

$$\begin{aligned} -\kappa = & \left(\Delta \sin^2 \theta \right)^{-2/(2\omega+3)} \left[- \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3} r^2 \right) c^2 \left(\frac{dt}{d\tau} \right)^2 + r^2 \sin^2 \theta \left(\frac{d\phi}{d\tau} \right)^2 \right] \\ & + \left(\Delta \sin^2 \theta \right)^{2/(2\omega+3)} \left[\left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3} r^2 \right)^{-1} \left(\frac{dr}{d\tau} \right)^2 + r^2 \left(\frac{d\theta}{d\tau} \right)^2 \right]. \end{aligned} \quad (34)$$

where $\kappa = c^2$ for timelike geodesics (i.e., for massive bodies) and $\kappa = 0$ for null geodesics (i.e., for light rays). Now, one may use eqs.(32) and (33) to eliminate $(dt/d\tau)$ and $(d\phi/d\tau)$ in terms of \tilde{E} and \tilde{L} and the result may be substituted into eq.(34) in order to obtain the “first integral” of the radial geodesic equation. Further, since gases or stars orbiting around the host galaxy center usually lie on the galactic (equatorial) plane, the case of equatorial geodesics at $\theta = \pi/2$ is of particular interest and the result is

$$\left(\frac{dr}{d\tau} \right)^2 + V_{eff}(\tilde{E}, \tilde{L}; r) = 0, \quad (35)$$

$$V_{eff}(\tilde{E}, \tilde{L}; r) = \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right) \left[\frac{\tilde{L}^2}{r^2} + \kappa\Delta^{-2/(2\omega+3)}\right] - \frac{\tilde{E}^2}{c^2}.$$

Therefore, now the problem of obtaining the timelike and null geodesics on the equatorial plane of the Schwarzschild-de Sitter-type spacetime in BD theory reduces to solving a problem of non-relativistic, one-dimensional motion in an effective potential $V_{eff}(\tilde{E}, \tilde{L}; r)$. Here we consider the case of our particular interest, the “stable” circular orbit motions of massive objects characterized by the simultaneous conditions of

$$V_{eff} = 0, \quad \frac{dV_{eff}}{dr} = 0 \quad (36)$$

as the ones first studied by Bardeen, Press and Teukolsky [19] for the case of Kerr spacetime. Then the conditions for the stable circular orbit motions in eq.(36) yield the required values of the specific energy and the specific angular momentum as

$$\tilde{E}^2 = c^2 \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right) \left[\frac{\tilde{L}^2}{r^2} + c^2\Delta^{-2/(2\omega+3)}\right], \quad (37)$$

$$\tilde{L}^2 = \frac{c^2 r^2}{(1 - 3\tilde{M}/r)} \left[\left(\frac{\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right) - \frac{2}{(2\omega+3)} \left(1 - \frac{\tilde{M}}{r} - \frac{16\pi\tilde{\Lambda}}{3}r^2\right) \right] \Delta^{-2/(2\omega+3)}. \quad (38)$$

Note that it is the rotation angular velocity (per unit test mass) v_ϕ rather than the specific angular momentum that we are after. Thus consider, again on the equatorial plane, the *proper* rotation velocity given by $v_\phi^2 = (d\sigma_3/dt)^2$ with $ds^2 = g_{tt}dt^2 + d\sigma_3^2$ (i.e., $d\sigma_3^2$ denotes *spatial* line element) and $dr = d\theta = 0$, namely

$$\begin{aligned} v_\phi^2 &= g_{\phi\phi} \left(\frac{d\phi}{dt}\right)^2 = g_{\phi\phi} \frac{(d\phi/d\tau)^2}{(dt/d\tau)^2} \\ &= \Delta^{-2/(2\omega+3)} r^2 \frac{\tilde{L}^2/r^4}{(1 - 2\tilde{M}/r - 8\pi\tilde{\Lambda}r^2/3)^{-2} \tilde{E}^2/c^4} \end{aligned} \quad (39)$$

where we used eqs.(32) and (33). Further, substituting eqs.(37) and (38) into eq.(39) yields

$$\begin{aligned} v_\phi^2 &= c^2 \Delta^{-2/(2\omega+3)} \times \\ &\frac{(1 - 2\tilde{M}/r - 8\pi\tilde{\Lambda}r^2/3) \left\{ (\tilde{M}/r - 8\pi\tilde{\Lambda}r^2/3) - \frac{2}{(2\omega+3)} (1 - \tilde{M}/r - 16\pi\tilde{\Lambda}r^2/3) \right\}}{(1 - 3\tilde{M}/r) + \left\{ (\tilde{M}/r - 8\pi\tilde{\Lambda}r^2/3) - \frac{2}{(2\omega+3)} (1 - \tilde{M}/r - 16\pi\tilde{\Lambda}r^2/3) \right\}}. \end{aligned} \quad (40)$$

This is the fully relativistic expression for the rotation angular velocity (per unit test mass). Lastly, in the outer region of BD scalar halo, where $r \sim 10 \text{ (kpc)} \simeq 10^{23} \text{ (cm)} \gg G_0 M/c^2 \simeq 10^{16} \text{ (cm)}$ and hence $\Delta = [r^2(1 - 8\pi\tilde{\Lambda}r^2/3) - 2\tilde{M}r]/r_0^2 \simeq [r^2(1 - 8\pi\tilde{\Lambda}r^2/3)]/r_0^2$, this fully

relativistic expression for the rotation angular velocity reduces to

$$v_\phi^2 = c^2 \left[\frac{(1 - 8\pi\tilde{\Lambda}r^2/3)}{(1 - 8\pi\tilde{\Lambda}r^2/3) - \frac{2}{(2\omega+3)}(1 - 16\pi\tilde{\Lambda}r^2/3)} \right] \times \left[\frac{8\pi\tilde{\Lambda}}{3}r^2 + \frac{2}{(2\omega+3)} \left(1 - \frac{16\pi\tilde{\Lambda}}{3}r^2 \right) \right] \left[\frac{r^2}{r_0^2} \left(1 - \frac{8\pi\tilde{\Lambda}}{3}r^2 \right) \right]^{-2/(2\omega+3)}. \quad (41)$$

Further, upon substituting a large- ω value, say, $\omega \sim 10^6$, one ends up with

$$\begin{aligned} v_\phi^2 &\simeq c^2 \left[\frac{2}{(2\omega+3)} + \frac{(2\omega-1)}{(2\omega+3)} \frac{8\pi\tilde{\Lambda}}{3}r^2 \right] \left\{ \frac{r^2}{r_0^2} \left(1 - \frac{8\pi\tilde{\Lambda}}{3}r^2 \right) \right\}^{-2/(2\omega+3)} \\ &= \left[100(km/s) \times \left\{ \frac{r^2}{r_0^2} \left(1 - \frac{8\pi G_0}{3c^4}\Lambda r^2 \right) \right\}^{-(1/10^6)} \right]^2 \\ &\quad + \left[c^2 \left(\frac{8\pi G_0}{3c^4}\Lambda r_0^2 \right) \left(\frac{r}{r_0} \right)^2 \right] \left\{ \frac{r^2}{r_0^2} \left(1 - \frac{8\pi G_0}{3c^4}\Lambda r^2 \right) \right\}^{-(1/10^6)} \end{aligned} \quad (42)$$

where as before the normalization factor $r_0 \sim 10$ (kpc) $\simeq 10^{23}$ (cm) which represent a typical distance to the outer region of the dark halo has been introduced. As has been discussed in the text, therefore, in order for this result to successfully describe the flattened rotation curve in the outer region of dark halo, $r \geq r_0$, it should be imposed that

$$\left(\frac{8\pi G_0 \Lambda}{3c^4} r_0^2 \right) \ll 10^{-6}$$

which again yields $\Lambda \ll 1.45 \times 10^{-4}$ (erg/cm³). Note that this is essentially the same result as what we have gotten using the Newtonian mechanics version of the rotation velocity in section IV. To summarize, even if we employ the fully relativistic expression for the rotation velocity, the conclusions we have reached in the previous section essentially remain the same.

B. Singularity along the Symmetry Axis: the Relativistic Bipolar Jet Interpretation

Earlier in section II, we mentioned in advance that a rigorous study of the true nature of the singularity along the symmetry axis reveals the fact that the bizzare singularity at $\theta = 0, \pi$ of the Schwarzschild-de Sitter-type solution in BD gravity theory can account for the relativistic bipolar outflows (twin jets) extending from the central region of “active galactic nuclei (AGNs)”. Thus in the present section, we shall demonstrate in an explicit manner that this is indeed the case.

In order to see if indeed the Schwarzschild-de Sitter-type spacetime solution in BD gravity theory allows for the occurrence of relativistic bipolar outflows, we particularly explore the geodesic motion of a test particle in the immediate vicinity of the symmetry axis, $\theta = 0, \pi$. Namely, consider the geodesic motion at $\theta \simeq \delta \ll 1$, (and hence $\sin \theta \simeq \sin \delta \equiv \epsilon \ll 1$). Then the conserved quantities, i.e., the specific energy $\tilde{E} = E/m$ and the specific angular momentum $\tilde{L} = L/m$ of a test particle along the geodesic are now given by

$$\begin{aligned}\tilde{E} &= (\epsilon^2 \Delta)^{-2/(2\omega+3)} \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3} r^2 \right) c^2 \left(\frac{dt}{d\tau} \right) \gg 1, \\ \tilde{L} &= (\epsilon^2 \Delta)^{-2/(2\omega+3)} r^2 \epsilon^2 \left(\frac{d\phi}{d\tau} \right) = (\epsilon^2)^{(2\omega+1)/(2\omega+3)} \Delta^{-2/(2\omega+3)} r^2 \left(\frac{d\phi}{d\tau} \right) \ll 1.\end{aligned}\tag{43}$$

Namely, the specific energy of a test particle is extremely large and its specific angular momentum is very small there for generic ω -value. And it is rather obvious to see that mainly this can be attributed to the metric function factor $(\Delta \sin^2 \theta)^{-2/(2\omega+3)}$ (where $\sin \theta$ is replaced by $\epsilon \ll 1$) which is responsible for the singular nature of the symmetry axis. Note first that the extremely small specific angular momentum near the symmetry axis is rather expected (like in the Einstein gravity context) as it is basically due to the negligible arm length from the symmetry axis. (Angular velocity $(d\phi/d\tau)$ there, however, may not be so small.) The extremely large specific energy near the symmetry axis, however, is indeed something unexpected and thus is surprising as it does not happen in the Einstein gravity context (i.e., the $\omega \rightarrow \infty$ limit) where the symmetry axis is perfectly regular. Now, this study of the true nature of the singularity along the symmetry axis leads us to suspect that perhaps the bizzare singularity at $\theta = 0, \pi$ of the Schwarzschild-de Sitter-type solution in BD gravity theory can account for the relativistic *bipolar outflows (twin jets)* extending from the central region of active galactic nuclei (AGNs). That is, the curious singularity along the symmetry axis seems harmless, after all. Rather, it turns out to be a pleasant surprise as it can explain a long-known puzzle in observed features of galaxies. We now demonstrate that indeed this extremely large specific energy near the symmetry axis can be translated into the ultrarelativistic speed at which the test particle moves along the symmetry axis. To this end, we start with $g_{\mu\nu} u^\mu u^\nu = -\kappa$ particularly near the poles, namely,

$$\begin{aligned}-\kappa &= (\epsilon^2 \Delta)^{-2/(2\omega+3)} \left[- \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3} r^2 \right) c^2 \left(\frac{dt}{d\tau} \right)^2 + r^2 \epsilon^2 \left(\frac{d\phi}{d\tau} \right)^2 \right] \\ &+ (\epsilon^2 \Delta)^{2/(2\omega+3)} \left[\left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3} r^2 \right)^{-1} \left(\frac{dr}{d\tau} \right)^2 + r^2 \left(\frac{d\delta}{d\tau} \right)^2 \right]\end{aligned}$$

$$\begin{aligned} &\simeq -(\epsilon^2 \Delta)^{-2/(2\omega+3)} \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right) c^2 \left(\frac{dt}{d\tau}\right)^2 \\ &+ (\epsilon^2 \Delta)^{2/(2\omega+3)} \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 \end{aligned} \quad (44)$$

where we used $\delta \ll 1$ and $\epsilon \ll 1$. We are now ready to study the first integral of the timelike (for massive particles, $\kappa = c^2$) radial geodesic equation given by

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= (\epsilon^2 \Delta)^{-4/(2\omega+3)} \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right) c^2 \left[\left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right) \left(\frac{dt}{d\tau}\right)^2 - (\epsilon^2 \Delta)^{2/(2\omega+3)} \right] \\ &\simeq (\epsilon^2 \Delta)^{-4/(2\omega+3)} \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right)^2 c^2 \left(\frac{dt}{d\tau}\right)^2. \end{aligned} \quad (45)$$

Thus along the symmetry axis $\theta = 0, \pi$, the *proper* ejection velocity would be given by

$$\begin{aligned} v_r^2 &= g_{rr} \left(\frac{dr}{dt}\right)^2 = g_{rr} \frac{(dr/d\tau)^2}{(dt/d\tau)^2} \\ &= c^2 \frac{1}{[\epsilon^2(r/r_0)^2]^{2/(2\omega+3)}} \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right)^{(2\omega+1)/(2\omega+3)} \\ &\simeq c^2 \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right) \end{aligned} \quad (46)$$

where we used eq.(45) and $\Delta = [r^2(1 - 8\pi\tilde{\Lambda}r^2/3) - 2\tilde{M}r]/r_0^2$ in the second line and $\omega \sim 10^6 \sim$ large in the last line. Once again, it is interesting to note that if the value of the BD ω -parameter were *not* large enough (like $\omega \sim 10^6$ which has been assigned to reproduce the flattened rotation curve in the galactic halos), the ejection speed of test particles along the symmetry axis (dr/dt) could encounter the danger of exceeding the speed of light due to the factor $1/(\epsilon^2)^{2/(2\omega+3)}$ on the right hand side for (dr/dt) . Lastly, for a typical galaxy with total mass $M \sim 10^{11}M_\odot$ and with the observed value of the cosmological constant $\Lambda \simeq 10^{-8} \text{ (erg/cm}^3\text{)}$, we have $2G_0M/c^2 \simeq 0.01 \text{ pc}$ and $(8\pi G_0\Lambda/3c^4)^{-1/2} \simeq 4 \text{ Gpc}$ and hence we end up with

$$v_r \simeq c \left(1 - \frac{2\tilde{M}}{r} - \frac{8\pi\tilde{\Lambda}}{3}r^2\right)^{1/2} = c \left[1 - \frac{(0.01 \text{ pc})}{r} - \left(\frac{r}{4 \times 10^9 \text{ pc}}\right)^2\right]^{1/2}. \quad (47)$$

Observationally, it is well-known that the typical size of the active galactic nuclei is less than 1 (pc) and the extent of typical galactic jets ranges from several (kpc) to a few (Mpc). Thus the ejection speed of test particles along the symmetry axis (i) just outside the AGN, i.e., for $r \sim \text{few (pc)}$ is $v_r \simeq c[1 - 0.01]^{1/2} \simeq 0.995c$ and (ii) well-above the galactic plane, say, for $r \sim$

10 (kpc) is $v_r \simeq c[1 - 10^{-6} - 10^{-10}]^{1/2} \leq c$. This completes the rigorous demonstration that the test particles move along the symmetry axis at nearly the speed of light. In addition, since the present analysis of the geodesic motion is valid only in the immediate vicinity of the symmetry axis, it indicates the *collimation* of the outflow as well consistently with the observation (see Fig.2).



FIG. 2: Centaurus A : An elliptical galaxy with an AGN and a jet.

VI. CONCLUDING REMARKS

In the present work, the pure (i.e., with no ordinary matter) BD gravity with or without the cosmological constant Λ has been demonstrated as a successful candidate for the theory of dark matter. To summarize, the mysterious flattened rotation curves observed for so long in the outer region of galactic halos, among others, have been the primary evidence for the existence of dark matter. Therefore, it has been demonstrated in this work that our model theory can successfully predict the emergence of dark matter halo-like configuration in terms of a self-gravitating static and nearly spherically-symmetric spacetime solution to the BD field equations and reproduce the flattened rotation curve in the outer region of this dark halo-like object in terms of the non-trivial energy density of the BD scalar field which was absent in the context of general relativity where the Newton's constant is strictly

a “constant” having no dynamics. As stated earlier, there has been a consensus that unless one modifies either the general relativity, the standard theory of gravity, or the standard model for particle physics, or both, one can never attain the satisfying understanding of the phenomena associated with dark matter and dark energy. Therefore, our dark matter model presented in this work can be viewed as an attempt to modify the gravity side alone in terms of the Brans-Dicke theory to achieve the goal. After all, the interesting lesson we learned from this study is the fact that the simplest extension of general relativity which involves relaxing Newton’s constant to a dynamical field (i.e., the BD scalar field $\Phi(x)$) essentially makes all the differences. Thus to conclude, from this success of “BD scalar field as the dark matter” to account for the asymptotic flattening of galaxy rotation curves while forming galactic dark matter halos *plus* the original spirit of the BD theory in which the BD scalar field is prescribed *not* to have direct interaction with ordinary matter fields (in order not to interfere with the great success of equivalence principle), we suggest that the Brans-Dicke theory of gravity *is* a very promising theory of dark matter. And this implies, if we emphasize it once again, that dark matter (and dark energy as well, see [4]) might not be some kind of an unknown exotic “matter”, but instead the effect resulting from the space-time varying nature of the Newton’s constant represented by the BD scalar field. Even further, this successful account of the phenomena associated with dark matter of the present universe via the BD gravity theory might be an indication that the truly relevant theory of classical gravity at the present epoch is not general relativity but its simplest extension, the Brans-Dicke theory with its generic parameter value $\omega \sim 10^6$ fixed by the dark matter observation. This idea, however, should be taken with some caution as there are other phenomena observed thus far (than the flattened rotation curves) which are suspected to be related to the effects of dark matter. These include the galaxy/cluster lensing of distant quasars. Besides, our model theory for dark matter presented in this work is not without a flaw. Earlier, from the rigorous study of the true nature of the singularity along the symmetry axis of this Schwarzschild-de Sitter-type spacetime, we suspected that perhaps this bizzare singularity at $\theta = 0, \pi$ can account for the relativistic bipolar outflows (twin jets) extending from the central region of active galactic nuclei (AGNs). The relativistic bipolar outflows, however, have been observed only for some particular types of galaxies such as AGNs and micro-quasars. That is, the jets do not seem to be a general feature of all types of galaxies. Therefore the Schwarzschild-de Sitter-type solution in BD gravity

theory employed in the present work appears to come with only a limited descriptive power for the observed phenomena associated with galaxies. At present in the absence of Birkhoff-type theorem guaranteeing the uniqueness of the Schwarzschild-type solution in BD gravity theory, however, it seems worth looking for an alternative solution and proceeding with it toward the issue addressed in the present work.

Therefore more extensive and careful study needs to be done to test the Brans-Dicke theory as a truly successful model theory of dark matter and dark energy and we hope to report more along this line in the near future.

Acknowledgements

This work was supported in part by the Korea Research Council of Fundamental Science and Technology (KRCF), Grant No. C-RESEARCH-2006-11-NIMS.

Appendix : Halos of BD scalar field : the case of Einstein frame

Note that in the present work, we ignore the presence of other types of ordinary matters. Furthermore, we particularly consider the case when the cosmological constant term (which is a dimensionful parameter in the action) is neglected as well. Namely, we are dealing with the pure Brans-Dicke theory of gravity and we particularly restrict our interest to the original spirit of Brans and Dicke according to which the BD scalar field Φ is prescribed to remain strictly massless without any self-interaction potential. Then in such a context, the study of possible role of BD scalar field as a dark matter component in Einstein frame, which is related to the Jordan frame (which we have been working in) by the Weyl rescaling (i.e., conformal transformation plus field redefinition), would be relevant as well. Thus we shall turn to this issue in this appendix.

We now begin with the Weyl-rescaling given by

$$g_{\mu\nu} = \Omega^2(x) \tilde{g}_{\mu\nu}, \quad \Phi = M_{pl}^2 e^{\Psi/\Psi_0} \quad (48)$$

$$\text{with } \Omega^2(x) = \frac{M_{pl}^2}{\Phi}, \quad \Psi_0^2 = (2\omega + 3)$$

where $M_{pl}^2 = 1/G_0$ denotes the Planck mass. Under this Weyl-rescaling, the action in the Jordan frame given earlier in eq.(1) but without the Λ -term transforms to

$$\tilde{S} = \int d^4x \sqrt{\tilde{g}} \frac{M_{pl}^2}{16\pi} \left[\tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi \right]. \quad (49)$$

Then extremizing this action with respect to the metric $\tilde{g}_{\mu\nu}$ and the redefined BD scalar field Ψ yields the classical field equations given respectively by

$$\begin{aligned} \tilde{G}_{\mu\nu} &= \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = 8\pi \tilde{T}_{\mu\nu}^{BD}, \quad \tilde{g}^{\alpha\beta} \nabla_\alpha \nabla_\beta \Psi = 0 \quad \text{where} \\ \tilde{T}_{\mu\nu}^{BD} &= \frac{1}{16\pi} \left[\nabla_\mu \Psi \nabla_\nu \Psi - \frac{1}{2} \tilde{g}_{\mu\nu} (\tilde{g}^{\alpha\beta} \nabla_\alpha \Psi \nabla_\beta \Psi) \right]. \end{aligned} \quad (50)$$

The action and the classical field equations in this Einstein frame appear to take the forms of those of massless scalar field theory coupled minimally to Einstein gravity. A caution, however, needs to be exercised. That is, despite how it seems, this theory in the Einstein frame is really the (pure) BD gravity in disguise. Indeed, close inspection of eq.(48) above reveals that the Weyl rescaling becomes trivial only for $\omega \rightarrow \infty$, which is the Einstein gravity limit (with no scalar field), but for finite ω values this theory is just the conformal rescaling of the (pure) BD gravity. As such, the solution to these field equations in the Einstein frame in eq.(50) should not be looked for independently without referring to that in the Jordan frame. Indeed, it would simply be the *same* Weyl rescaling of the solution constructed in the Jordan frame given in eq.(14), namely,

$$\begin{aligned} d\tilde{s}^2 &= \tilde{g}_{\mu\nu} dx^\mu dx^\nu = \left(\frac{\Phi}{M_{pl}^2} \right) g_{\mu\nu} dx^\mu dx^\nu \\ &= \left[- \left(1 - \frac{2M}{r} \right) dt^2 + r^2 \sin^2 \theta d\phi^2 \right] + (\Delta \sin^2 \theta)^{4/(2\omega+3)} \left[\left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\theta^2 \right], \\ \Psi(r, \theta) &= \Psi_0 \ln (\Phi/M_{pl}^2) \\ &= 2(2\omega + 3)^{-1/2} \ln (\Delta \sin^2 \theta). \end{aligned} \quad (51)$$

We now attempt to address the same issues as we did earlier while working in the Jordan frame, i.e., whether the BD scalar field can cluster into dark matter halo-like objects and reproduce flattened rotation curves. Firstly, once again it is obvious that the BD scalar can cluster into halo-like configuration as it can be represented by this Brans-Dicke-Schwarzschild solution in the Einstein frame given in eq.(51). Secondly, we turn to the behavior of the rotation curve in the outer region of this halo-like configuration. To this end, again we

restore both the Newton's constant G_0 and the speed of light c to work in the CGS unit as before by multiplying the factor (c^4/G_0) to the BD scalar field energy-momentum tensor in eq.(50) and by replacing $M \rightarrow G_0 M/c^2 \equiv \tilde{M}$ in the BDS solution in Einstein frame given in eq.(51). Next, by substituting the BDS solution in eq.(51) into the BD scalar energy-momentum tensor in eq.(50) and then setting it equal to (15), again we can read off the energy density and the pressure components of the BD scalar field imperfect fluid as

$$\begin{aligned}\rho &= \frac{c^2}{2\pi G_0} \frac{1}{(2\omega+3)} (\Delta \sin^2 \theta)^{-4/(2\omega+3)} \left[\frac{(r-\tilde{M})^2}{r^2 \tilde{\Delta}} + \frac{\cos^2 \theta}{r^2 \sin^2 \theta} \right], \\ P_r &= \frac{c^4}{2\pi G_0} \frac{1}{(2\omega+3)} (\Delta \sin^2 \theta)^{-4/(2\omega+3)} \left[\frac{(r-\tilde{M})^2}{r^2 \tilde{\Delta}} - \frac{\cos^2 \theta}{r^2 \sin^2 \theta} \right], \\ P_\theta &= -P_r, \quad P_\phi = -c^2 \rho, \\ T_\theta^r &= \Delta T_r^\theta = \frac{c^4}{\pi G_0} \frac{1}{(2\omega+3)} (\Delta \sin^2 \theta)^{-4/(2\omega+3)} \frac{(r-\tilde{M})}{r^2} \cot \theta.\end{aligned}\tag{52}$$

Now, in order to study the behavior of the rotation curves in the outer region of dark halos of typical galaxies, we approximate the expression for the energy density of the BD scalar field at large- r , say, $r \gg G_0 M/c^2$ and it is $\rho \simeq c^2(2\pi G_0(2\omega+3))^{-1}(r^2 \sin^2 \theta)^{-1} (\Delta \sin^2 \theta)^{-4/(2\omega+3)}$. Then the mass function is given by $M(r) = \int_0^{2\pi} d\phi \int_\epsilon^{\pi-\epsilon} d\theta \int_0^r dr \sqrt{\tilde{g}_{rr}\tilde{g}_{\theta\theta}\tilde{g}_{\phi\phi}} \rho(r, \theta) = (c^2/G_0(2\omega+3)) f(\delta) r$. Note here that we already have $M(r) \sim r$ regardless of the value of BD ω -parameter in the present case of Einstein frame and hence

$$v^2(r) = \frac{G_0 M(r)}{r} = c^2 \frac{1}{(2\omega+3)} f(\delta)\tag{53}$$

where this time $f(\delta) \equiv \int_\epsilon^{\pi-\epsilon} d\theta \sin^{-1} \theta = 2 \int_0^{1-\delta} dx [1-x^2]^{-1} \simeq O(1)$ with $\epsilon, \delta \ll 1$. This is already a flattened rotation curve, and if we further take, say, $\omega \sim 10^6$ once again, we get

$$v(r) \simeq 100(km/s).\tag{54}$$

Therefore, regardless of whether we work in the Jordan or in the Einstein frame, the BD scalar field imperfect fluid always reproduces the flattened rotation curve. And from this observation, the BD scalar field appears to play the role of dark matter component in the galactic halos.

Consider next, the equation of state of this BD scalar field imperfect fluid in this Einstein frame

$$w = \frac{P}{c^2 \rho} = \frac{[(r-\tilde{M})^2/r^2 \tilde{\Delta}] - [\cos^2 \theta / r^2 \sin^2 \theta]}{[(r-\tilde{M})^2/r^2 \tilde{\Delta}] + [\cos^2 \theta / r^2 \sin^2 \theta]}\tag{55}$$

where $P = P_r$. Just like in the earlier study in the Jordan frame, $P = w(r, \theta)c^2\rho$ with $w(r, \theta) \sim O(1)$ meaning that this BD scalar fluid is still a barotropic fluid but with position-dependent coefficient $w(r, \theta)$. And once again, although the BD scalar field is a candidate for dark matter, this equation of state indicates that in general, particularly in the vicinity of the individual galaxy, it is not quite a dust. In the outer halo region, i.e., at large- r , however, $w \simeq 0$ (as $P \simeq 0$ in the numerator) meaning that indeed the BD scalar fluid in the Einstein frame exhibits the property of *dust* matter everywhere including on the galactic (equatorial) plane regardless of the value of BD parameter ω .

We now compare the results obtained earlier in the Jordan frame with those here in the Einstein frame. Firstly, in both frames, the BD scalar field imperfect fluid appears to play the role of dark matter component in a successful manner as it always reproduces the flattened rotation curve in the outer region of galactic halos. Secondly, the equations of state of the BD scalar fluid in both frames indicate that it is generally not quite a dust and this point implies that the BD scalar field appears to be an “exotic” type of dark matter possessing non-standard equation of state generally in the vicinity of individual galaxy. In the outer halo region, i.e., at large- r , however, it is interesting to note that $w \simeq 0$, namely a dust matter for large- ω parameter value and particularly on the galactic (equatorial) plane, where most of the gases or stars orbit around the galaxy center, in the case of Jordan frame whereas it is $w \simeq 0$ everywhere, *not* just on the galactic (equatorial) plane, and *regardless* of the value of ω in the case of Einstein frame.

One might wonder how this somewhat delicate discrepancy in the nature of equation of state for the BD scalar fluid in the two conformal frames arises. We now attempt to provide one possible cause that may lead to this slight discrepancy. Indeed, close inspection reveals that it can be attributed to the non-standard form of the BD scalar field energy-momentum tensor in the Jordan frame. That is, in the expression for the BD scalar field energy-momentum tensor given in eq.(2), we can realize that it involves both the terms quadratic in first derivatives of the BD scalar field Φ coming from its kinetic energy term in the action and those linear in second derivatives of Φ coming from its non-minimal coupling term to curvature, $\sim \sqrt{g}\Phi R$. (Note that its standard expression for the ordinary scalar field (like that in the Einstein frame given in eq.(50) for the Weyl rescaled BD scalar field Ψ) involves only the terms quadratic in first derivatives of the scalar field.) And these *anomalous* terms linear in second (covariant) derivatives of Φ seem to be the ones that make the difference as

they turn out to render the pressure of the BD scalar imperfect fluid *negative* at large- r and particularly for large- ω while *zero* on galactic equatorial plane in the Jordan frame. In both frames, however, the BD scalar field imperfect fluid appears to reproduce generic features of the dark matter in the outer region of dark halo of the individual galaxy.

Note that in the other case where the cosmological constant is present, the issue of selecting from Jordan or Einstein frame that we just discussed becomes irrelevant to address. And it is because when a dimensionful parameter such as the cosmological constant (with mass dimension 4 in the Planck unit) is present in the theory action, the Weyl-rescaling of the action/field equations to go from the original Jordan frame, say, to Einstein frame does not leave the physics invariant. Therefore in section IV in the text, we just worked in the original Jordan frame.

References

- [1] C. Brans and C. H. Dicke, Phys. Rev. **124**, 925 (1961).
- [2] C. M. Will, *Was Einstein Right ?*, (Basic Books, Inc., Publishers/New York, 1986).
- [3] S. Weinberg, *Gravitation and Cosmology*, (John Wiley and Sons, Inc., 1972).
- [4] H. Kim, Phys. Lett. **B606**, 223 (2005) ; Mon. Not. Roy. Astron. Soc. **364**, 813 (2005).
- [5] R. N. Tiwari and B. K. Nayak, Phys. Rev. **D14**, 2502 (1976) ; J. Math. Phys. **18**, 289 (1977).
- [6] T. Singh and L. N. Rai, Gen. Rel. Grav. **11**, 37 (1979).
- [7] R. A. Matzner and C. W. Misner, Phys. Rev. **154**, 1229 (1967).
- [8] R. M. Misra and D. B. Pandey, J. Math. Phys. **13**, 1538 (1972).
- [9] H. Kim, Phys. Rev. **D60**, 024001 (1999).
- [10] H. Kim and H. M. Lee, Int. J. Mod. Phys. A20, 6461 (2005).
- [11] F. E. Schunck and E. W. Mielke, Class. Quantum Grav. **20**, R301 (2003).
- [12] See for instance, over 100 rotation curve fits given in J. R. Brownstein and J. W. Moffat, Astrophys. J. **636**, 721 (2006) ; see also, V. Sahni, Lect. Notes Phys. 653, 141 (2004).
- [13] S. McGaugh, V. Rubin, and E. de Block, Astron. J. **122**, 2381 (2001).

- [14] M. Persic, P. Salucci, and F. Stel, *Mon. Not. Roy. Astron. Soc.* **281**, 27 (1996) ; E. Corbelli and P. Salucci, *astro-ph/9909252* ; Y. Sofue and V. Rubin, *Ann. Rev. Astron. Astrophys.* 39, 137 (2001).
- [15] R. B. Tully and J. R. Fisher, *Astron. Astrophys.* **54**, 661 (1977).
- [16] C. Armendariz-Picon and E. A. Lim, *JCAP* 0508, 007 (2005) ; F. E. Schunk, *astro-ph/9802258*.
- [17] C. L. Bennett et al., *Astrophys. J. Suppl.* **148**, 1 (2003) ; G. Hinshaw et al., *Astrophys. J. Suppl.* **148**, 135 (2003) ; D. N. Spergel et al., *Astrophys. J. Suppl.* **148**, 175 (2003).
- [18] E. J. Copeland, M. Sami, and S. Tsujikawa, *hep-th/0603057*.
- [19] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, *Astrophys. J.* **178**, 347 (1972)